# A Note on Non–Additive Quantum Codes

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A method to combine two quantum error–correcting codes is presented. Even when starting with additive codes, the resulting code might be non–additive. Furthermore, the notion of the erasure space is introduced which gives a full characterisation of the erasure–correcting capabilities of the codes. For the special case that the two codes are unitary images of each other, the erasure space and the pure erasure space of the resulting code can be calculated.
**Note:** This report is preliminary, any suggestions or comments on errors are welcome.

## I. INTRODUCTION

In a very recent paper [5], Rains *et al* presented the first non–additive quantum error–correcting code. The code was constructed using numerical iteration to build a projector with a given weight distribution. That code was transformed into a code with orthogonal basis $\{|c_0\rangle, \ldots, |c_5\rangle\}$ where each of the basis elements is a $((5,1,3))$ additive quantum code. Futhermore, all of them are equivalent. For $i = 1, \ldots, 5$ the basis elements $|c_i\rangle$ can be obtained by multiplication of $|c_0\rangle$ with the transformation $\tau = \mathbb{1} \otimes \mathbb{1} \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x$ and its cyclic shifts.

This generalizes to the construction of *union quantum codes*, i. e., the code is the sum of the subspaces generated by two quantum codes of same length. The name union is motivated by the analogue construction for classical codes. Given two (linear) codes of equal length, the union of the first code and a properly chosen coset of the second yields a possibly non–linear code.

## II. QUANTUM CODES AND ERASURE SPACES

### A. General Codes

Following [4], let $\mathcal{C} = ((N, K, d))$ denote a quantum error–correcting code that spans a $K$-dimensional subspace of a $2^N$-dimensional Hilbert space. The code can correct any error that affects less than $\frac{d}{2}$ qubits, or equivalently, as shown in [2], it can correct erasures of up to $d - 1$ qubits.

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Recall that necessary and sufficient conditions for a quantum code with orthogonal basis $\{|c_i\rangle\}$ to correct up to $d - 1$ erasures are

$$\forall i \neq j : \langle c_i|E|c_j\rangle = 0 \tag{1}$$

$$\forall i, j : \langle c_i|E|c_i\rangle = \langle c_j|E|c_j\rangle \tag{2}$$

for any error operator $E$ of weight less than $d - 1$ (cf. [3] for the definition of error operators and [4] for the definition of their weight).

It is sufficient to consider only operators that are of the form $E = \sigma_{i_1} \otimes \ldots \otimes \sigma_{i_N}$ where the $\sigma_i$ are the identity operator $\mathbb{1}$ or one of the Pauli matrices $\sigma_x$, $\sigma_y$, $\sigma_z$. The set of error operators of that type will be denoted by $\mathcal{EB}$ and is an algebra basis of the operators on the $2^N$–dimensional Hilbert space.

In general, conditions (1) and (2) may be fulfilled for some operators of weight greater then $d - 1$. This motivates the definition of the *erasure space* of a quantum code:

**Definition 1**
*The* erasure space $\mathcal{E}(\mathcal{C}) \subseteq \mathbb{C}^{2^N \times 2^N}$ *of a quantum code* $\mathcal{C}$ *is the set of operators $E$ that fulfil (1) and (2).*

Clearly, $\mathcal{E}(\mathcal{C})$ is closed under addition. Furthermore, $E^\dagger \in \mathcal{E}(\mathcal{C})$ for every $E \in \mathcal{E}(\mathcal{C})$. Thus, every element of a ($\mathbb{C}$–vector space) basis of $\mathcal{E}(\mathcal{C})$ can be written as Hermitian operator $E + E^\dagger$ or anti–Hermitian operator $E - E^\dagger$. This is summarized by the following lemma:

**Lemma 1** *The* erasure space $\mathcal{E}(\mathcal{C})$ *is a subspace of the* $\mathbb{C}$ *vector space* $\mathbb{C}^{2^N \times 2^N}$. *It possesses a basis consisting of Hermitian and anti–Hermitian operators of the form* $E + E^\dagger$ *and* $E - E^\dagger$.

A basis of the erasure space $\mathcal{E}(\mathcal{C})$ is called an *erasure basis* $\mathcal{EB}(\mathcal{C})$ and is denoted by $\mathcal{E}(\mathcal{C})$.
If for a fixed set of positions any erasure can be corrected, $\mathcal{EB}(\mathcal{C})$ contains all error–operators $E \in \mathcal{EB}$ that introduce errors at that positions. This is not the case when only some specific erasures (e. g. phase–erasures) can be corrected.

### B. Pure Codes

Ekert and Macchiavello [1] gave the following sufficient, but not necessary condition for codes to correct erasures $E$:

$$\forall i, j : \langle c_i|E|c_j\rangle = \begin{cases} \alpha \delta_{ij} & \text{for } E = \alpha\mathbb{1}, \, \alpha \in \mathbb{C} \\ 0 & \text{else.} \end{cases} \tag{3}$$

Similarly to Def. 1, this defines a set of operators.

**Definition 2**
*The* pure erasure space $\mathcal{E}_{\mathrm{pure}}(\mathcal{C})$ *is the set of operators $E$ that fulfil (3).*

Again, this set is closed under addition, and since (1) and (2) imply (3) we get the following:

**Lemma 2** *The pure erasure space is a subspace of the erasure space, i. e., $\mathcal{E}_{\mathrm{pure}}(\mathcal{C}) \le \mathcal{E}(\mathcal{C})$.*

According to the literature, codes fulfilling (3) for all error–operators $E$ of weight less than $d$ are called pure or non–degenerate $d-1$–erasure–correcting codes.

## III. UNION QUANTUM CODES

### A. Definition

**Definition 3**
*Let $\mathcal{C}^{(1)} = ((N_1, K_1, d_1))$ and $\mathcal{C}^{(2)} = ((N_2, K_2, d_2))$ be quantum error–correcting codes with orthogonal bases $B^{(1)} = \{|\boldsymbol{c}_1^{(1)}\rangle, \ldots, |\boldsymbol{c}_{K_1}^{(1)}\rangle\}$ and $B^{(2)} = \{|\boldsymbol{c}_1^{(2)}\rangle, \ldots, |\boldsymbol{c}_{K_2}^{(2)}\rangle\}$, respectively. W. l. o. g. we can assume $N_1 = N_2 = N$ and $\mathcal{C}^{(1)} \perp \mathcal{C}^{(2)}$. Then the* union code $\mathcal{C}$ *is the quantum code with basis $B = B^{(1)} \cup B^{(2)}$.*

Note that each of the codes $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$ are subcodes of $\mathcal{C}$ and they are disjoint since $\mathcal{C}^{(1)} \perp \mathcal{C}^{(2)}$. This yields the following lemma.

**Lemma 3** *The union code $\mathcal{C}$ is a $((N, K, d))$ code with $K = K_1 + K_2$, $d \le \min(d_1, d_2)$, and $\mathcal{C} = \mathcal{C}^{(1)} \oplus \mathcal{C}^{(2)}$.*

(Here $\oplus$ denotes the sum of complex vector spaces, and the sum is direct since the codes are orthogonal.)

### B. The Erasure Space of Union Codes

#### 1. General Case

The union code $\mathcal{C}^{(1)} \oplus \mathcal{C}^{(2)}$ can correct all erasures $E$ that fulfil the following conditions:

$$\forall (\mu, i) \ne (\nu, j) : \langle \boldsymbol{c}_i^{(\mu)}|E|\boldsymbol{c}_j^{(\nu)}\rangle = 0 \tag{4}$$

$$\forall \mu, \nu, i, j : \langle \boldsymbol{c}_i^{(\mu)}|E|\boldsymbol{c}_i^{(\mu)}\rangle = \langle \boldsymbol{c}_j^{(\nu)}|E|\boldsymbol{c}_j^{(\nu)}\rangle \tag{5}$$

Since for $\mu = \nu$ these conditions are exactly those for the erasure spaces $\mathcal{E}(\mathcal{C}^{(1)})$ and $\mathcal{E}(\mathcal{C}^{(2)})$ we have

$$\mathcal{E}(\mathcal{C}^{(1)} \oplus \mathcal{C}^{(2)}) \le \mathcal{E}(\mathcal{C}^{(1)}) \cap \mathcal{E}(\mathcal{C}^{(2)}). \tag{6}$$

For $\mu \ne \nu$, we have the following additional constraints:

$$\forall i, j : \langle \boldsymbol{c}_i^{(1)}|E|\boldsymbol{c}_j^{(2)}\rangle = 0 \tag{7}$$

$$\forall i, j : \langle \boldsymbol{c}_i^{(2)}|E|\boldsymbol{c}_j^{(1)}\rangle = 0 \tag{8}$$

$$\langle \boldsymbol{c}_1^{(1)}|E|\boldsymbol{c}_1^{(1)}\rangle = \langle \boldsymbol{c}_1^{(2)}|E|\boldsymbol{c}_1^{(2)}\rangle \tag{9}$$

(in (9) it is sufficient to consider only one pair with $\mu \ne \nu$ and fixed indexes $i$ and $j$). Due to these additional constraints, the erasure space $\mathcal{E}(\mathcal{C}^{(1)} \oplus \mathcal{C}^{(2)})$ is, in general, a proper subspace of $\mathcal{E}(\mathcal{C}^{(1)}) \cap \mathcal{E}(\mathcal{C}^{(2)})$.

#### 2. Codes of Equal Dimension

If the codes $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$ are of equal dimension, there is a unitary transformation $U \in U(2^N)$ such that $\mathcal{C}^{(2)} = U\mathcal{C}^{(1)}$. Then each basis element of $\mathcal{C}^{(2)}$ can be written as $|\boldsymbol{c}_i^{(2)}\rangle = U|\boldsymbol{c}_i^{(1)}\rangle$. Using $\langle \boldsymbol{c}_i^{(2)}|E|\boldsymbol{c}_j^{(2)}\rangle = \langle \boldsymbol{c}_i^{(1)}U^\dagger|E|U\boldsymbol{c}_j^{(1)}\rangle$ yields that the error spaces are conjugated, i. e.,

$$\mathcal{E}(\mathcal{C}^{(2)}) = \mathcal{E}(U\mathcal{C}^{(1)}) = U\mathcal{E}(\mathcal{C}^{(1)})U^\dagger. \tag{10}$$

Henceforth we will drop the superscript $^{(1)}$. From (6) it follows that

$$\mathcal{E}(\mathcal{C} \oplus U\mathcal{C}) \le \mathcal{E}(\mathcal{C}) \cap U\mathcal{E}(\mathcal{C})U^\dagger. \tag{11}$$

Furthermore, conditions (7), (8), and (9) read

$$\forall i, j : \langle \boldsymbol{c}_i|EU|\boldsymbol{c}_j\rangle = 0 \tag{12}$$

$$\forall i, j : \langle \boldsymbol{c}_i|U^\dagger E|\boldsymbol{c}_j\rangle = 0 \tag{13}$$

$$\langle \boldsymbol{c}_1|E|\boldsymbol{c}_1\rangle = \langle \boldsymbol{c}_1|U^\dagger EU|\boldsymbol{c}_1\rangle \tag{14}$$

Since condition (12) and (13) imply (3) for the operators $E' = EU$ and $E'' = U^\dagger E$, resp., we have

$$\mathcal{E}(\mathcal{C} \oplus U\mathcal{C}) \le \mathcal{E}_{\mathrm{pure}}(\mathcal{C})U^\dagger \cap U\mathcal{E}_{\mathrm{pure}}(\mathcal{C}). \tag{15}$$

Combining (11) and (15) yields an upper bound for the erasure space of the union code. What is more, we can compute the erasure space of the union code:

**Theorem 4**
*Let $\mathcal{M}(|\boldsymbol{c}_1\rangle, U)$ be the vector space of operators $E$ with*

$$\langle \boldsymbol{c}_1|E|\boldsymbol{c}_1\rangle = \langle \boldsymbol{c}_1|U^\dagger EU|\boldsymbol{c}_1\rangle. \tag{16}$$

*The erasure space of the union code $\mathcal{C} \oplus U\mathcal{C}$ is given by*

$$\begin{aligned}
\mathcal{E}(\mathcal{C} \oplus U\mathcal{C}) = {} & \mathcal{E}(\mathcal{C}) \cap U\mathcal{E}(\mathcal{C})U^\dagger \\
& \cap \mathcal{E}_{\mathrm{pure}}(\mathcal{C})U^\dagger \cap U\mathcal{E}_{\mathrm{pure}}(\mathcal{C}) \\
& \cap \mathcal{M}(|\boldsymbol{c}_1\rangle, U).
\end{aligned} \tag{17}$$

**Proof:** The conjunction of the defining conditions for the spaces on the right hand side of (17) is exactly the condition for an operator $E$ to be in $\mathcal{E}(\mathcal{C} \oplus U\mathcal{C})$. ∎
The pure erasure space of $\mathcal{C} \oplus U\mathcal{C}$ can be computed directly from the pure erasure space of $\mathcal{C}$:

**Theorem 5**

$$\mathcal{E}(\mathcal{C})_{\text{pure}} \cap U\mathcal{E}(\mathcal{C})_{\text{pure}}U^{\dagger}$$
$$\cap\, \mathcal{E}_{\text{pure}}(\mathcal{C})U^{\dagger} \cap U\mathcal{E}_{\text{pure}}(\mathcal{C}) = \mathcal{E}_{\text{pure}}(\mathcal{C} \oplus U\mathcal{C}). \quad (18)$$

**Proof:** An element $E$ of the left hand side of (18) can be written as $E = E_1 = UE_2U^{\dagger} = UE_3 = E_4U^{\dagger}$ with $E_i \in \mathcal{E}_{\text{pure}}(\mathcal{C})$. This implies

$$\langle \boldsymbol{c}_i|E|\boldsymbol{c}_j\rangle = \langle \boldsymbol{c}_i|E_1|\boldsymbol{c}_j\rangle = 0, \quad (19)$$

$$\langle \boldsymbol{c}_i|U^{\dagger}EU|\boldsymbol{c}_j\rangle = \langle \boldsymbol{c}_i|E_2|\boldsymbol{c}_j\rangle = 0, \quad (20)$$

$$\langle \boldsymbol{c}_i|U^{\dagger}E|\boldsymbol{c}_j\rangle = \langle \boldsymbol{c}_i|E_3|\boldsymbol{c}_j\rangle = 0, \quad \text{and} \quad (21)$$

$$\langle \boldsymbol{c}_i|EU|\boldsymbol{c}_j\rangle = \langle \boldsymbol{c}_i|E_4|\boldsymbol{c}_j\rangle = 0 \quad (22)$$

which proofs that $E \in \mathcal{E}_{\text{pure}}(\mathcal{C} \oplus U\mathcal{C})$. Furthermore, conditions (19) – (22) are exactly those for the spaces on the left hand side of (18) which proofs equality. ∎
Theorems 4 and 5 characterize the erasure space and the pure erasure space as intersection of several spaces. The weight structure of these spaces yields bounds on the weight of the erasures that can be corrected by the union code.

### 3. Equivalent Codes

Now consider the case when the codes $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$ are *locally permutation equivalent*, i.e., $\mathcal{C}^{(2)} = \tau\mathcal{C}^{(1)}$ with $\tau = \pi T$ where $t$ is a local unitary transformation $T \in U(2)^{\otimes N}$ and $\pi$ is a permutation of the qubits. In that case, conjugation with $\tau$ is weight preserving whereas left and right multiplications, in general, change the weight. Thus, the minimum distance of the union code is mainly determined by the weight structure of $\tau\mathcal{E}_{\text{pure}}(\mathcal{C})$ and $\mathcal{E}_{\text{pure}}(\mathcal{C})\tau^{\dagger}$.

### IV. EXAMPLES

#### A. The Code of Rains *et al*

As mentioned before, the code of Rains *et al* can be constructed as a union code of six subcodes which are $\mathcal{C}^{(0)}$ generated by

$$|\boldsymbol{c}\rangle = |00000\rangle - (|00011\rangle)_{\text{cyc}} + (|00101\rangle)_{\text{cyc}} - (|01111\rangle)_{\text{cyc}}$$

and $\mathcal{C}^{(i)} = \pi^i\tau\mathcal{C}^{(0)}$ where $\tau = \mathbb{1}\otimes\mathbb{1}\otimes\sigma_x\otimes\sigma_x\otimes\sigma_x$ and $\pi$ is a cyclic shift of the five qubits. The pure erasure space $\mathcal{E}_{\text{pure}}(\mathcal{C}^{(0)})$ contains all error–operators of weight one and two. There are twenty error–operators of weight three ($E_1 = \mathbb{1}\otimes\mathbb{1}\otimes\sigma_y\otimes\sigma_z\otimes\sigma_y$, $E_2 = \mathbb{1}\otimes\sigma_z\otimes\mathbb{1}\otimes\sigma_x\otimes\sigma_x$ and their cyclic shifts) that are not in the pure erasure space.
The weight of the operators $\pi^i\tau\pi^jE_1$ is at least three whereas the operators $\pi^i\tau\pi^jE_1$ include the operators

$$\sigma_x \otimes \sigma_z \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \qquad \sigma_z \otimes \sigma_x \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1},$$

and their cyclic shifts of weight two. Indeed these operators are not in $\mathcal{E}_{\text{pure}}(\mathcal{C})\tau$ and according to (17) not in $\mathcal{E}(\mathcal{C})$. Thus, the union code has minimum distance less than three. The minimum distance is two since $\mathcal{E}(\mathcal{C})$ contains all error–operators of weight one and there twenty error–operators of weight two that are not in $\mathcal{E}(\mathcal{C})$:

$$\sigma_x \otimes \sigma_z \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \qquad \sigma_z \otimes \sigma_x \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1},$$
$$\sigma_z \otimes \mathbb{1} \otimes \sigma_y \otimes \mathbb{1} \otimes \mathbb{1}, \qquad \sigma_y \otimes \mathbb{1} \otimes \sigma_z \otimes \mathbb{1} \otimes \mathbb{1},$$

and their cyclic shifts.

#### B. A Special Erasure Code

We start with the $((4, 4, 2))$–code presented in [2] that can corrrect one general erasure. This code $\mathcal{C}^{(1)}$ is generated by

$$\begin{aligned}
|\boldsymbol{c}_0\rangle &= |0000\rangle + |1111\rangle \\
|\boldsymbol{c}_1\rangle &= |0110\rangle + |1001\rangle \\
|\boldsymbol{c}_2\rangle &= |0101\rangle + |1010\rangle \\
|\boldsymbol{c}_3\rangle &= |1100\rangle + |0011\rangle
\end{aligned}$$

The local transformation $\tau = \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \sigma_y$ yields an equivalent code $\mathcal{C}^{(2)} = \tau\mathcal{C}^{(1)}$ generated by

$$\begin{aligned}
|\boldsymbol{c}_4\rangle &= |0001\rangle - |1110\rangle \\
|\boldsymbol{c}_5\rangle &= |0010\rangle - |1101\rangle \\
|\boldsymbol{c}_6\rangle &= |0100\rangle - |1011\rangle \\
|\boldsymbol{c}_7\rangle &= |1000\rangle - |0111\rangle.
\end{aligned}$$

The union code $\mathcal{C}^{(1)} \oplus \mathcal{C}^{(2)} = \mathcal{C}^{(1)} \oplus \tau\mathcal{C}^{(1)}$ is a $((4,8,1))$ code. It has minimum distance one and cannot correct a general erasure, but it can correct all phase–only or amplitude–only erasures. The reason for this is that $\mathcal{E}(\mathcal{C}^{(1)} \oplus \tau\mathcal{C}^{(1)})$ contains all error–operators of weight one of the form $\sigma_x \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}$ and $\sigma_z \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}$ and their cyclic shifts.

### V. CONCLUSION

Motivated by the classical analogue to join the sets of codewords of two codes to obtain a new one, the construction of union quantum codes has been introduced simultaneously encompassing special examples of known and new code. The the new notion of erasure spaces gives more insight in the erasure–correcting — and thereby error–correcting — capabilities of a code than just the minimum distance. The erasure space also provides a means to compare non–equivalent codes with the same weight distribution.

### VI. ACKNOWLEDGEMENTS

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